Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On Adjustable Cycloidal and Trochoidal Curves.

BY FRANK MORLEY.

There are certain well-known facts about the simpler cycloids,* such as the following:

A hypocycloid of class 3 makes a constant intercept on any tangent.

A cardioid makes a constant intercept on any line through the cusp.

The cusp-tangents of a hypocycloid of class 4 make an intercept of constant length on any tangent.

My object is to obtain some general theorems which include facts like those mentioned. A way of effecting this lies in the incidental remark of Chasles (Aperçu Historique, p. 69) that when a circle rolls on a fixed circle the envelope of a diameter of the rolling circle is a cycloid. Here we can regard the diameter as a hypocycloid of class 2 whose cusps are attached to the rolling circle; and the first extension of Chasles's remark is to ascertain the envelope of any cycloid whose cusps are attached to the rolling circle.

With the kinematical method by which Williamson (Differential Calculus, §291 of the fourth edition) establishes Chasles's result, it will be easy to prove that the envelope sought is two distinct cycloids. Some special cases of this general theorem will next be discussed, and the existence of mutually "adjustable" cycloids will present itself naturally. Lastly, the extended problem, when the moving curve is a trochoid, calls for consideration; a general theorem is given, but there is no discussion of the special cases; instead, the question of mutually "adjustable" trochoids is taken up *ab initio.*

It will be convenient to begin (§§1–3) with some remarks on the modes of generation of a trochoid and on the normals. So far as concerns the cycloidal case these remarks are practically already stated in a former paper, this Journal, t. XIII, in which the tangents of cycloids were briefly considered. But it is to be noticed that the normals of a trochoid, as of roulettes in general, are to be preferred to the tangents.

---

*It is convenient, and not without precedent, to include epicycloids and hypocycloids under the general term cycloid, and epitrochoids and hypotrochoids under the general term trochoid.

§1. The usual definition of a trochoid, as the roulette produced when a circle rolls on a fixed circle, is open to the objection that, of the two ways of regarding the curve as a roulette, one is arbitrarily preferred, the other subsequently dragged in. The ancient way of regarding the curve as an epicycle seems to me the proper one. It comes to this, that if we take a parallelogram 0, x'_q, x, x'_p (Fig. A), of which one point 0 is fixed, and two others, x'_p and x'_q, describe circles about 0 with assigned angular velocities p and q; then the locus of the fourth point x is defined as a trochoid.* If t is always a point of a unit circle, that is if $|t| = 1$, the equation which maps the unit circle into the trochoid is evidently

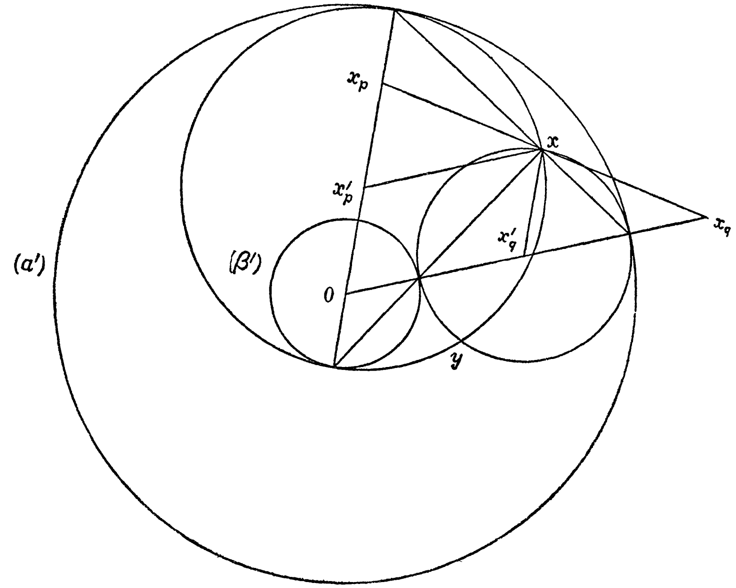

FIG. A.

$$x = \alpha t^q + \beta t^p, \tag{1}$$

where $\beta = |x_p|$, $\alpha = |x_q|$, and the parallelogram is supposed initially, when $t = 1$, to lie in a state of collapse along the positive half of the real axis.

---

[*] It is of slight use to talk of trochoids unless the curves are present. Mr. Henry Perigal's sheets of diagrams, called Contributions to Kinematics, are most serviceable here.

The way in which the definitions by means of rolling circles come out of this equation is indicated in Williamson, Differential Calculus, §295*a*, and in Harkness and Morley's Theory of Functions, p. 17, and the same results can also be obtained by considering the equation

$$dx = (q\alpha t^q + p\beta t^p)\,dt/t,$$

which, if $t = \exp i\theta$, is

$$dx \doteq i\,(q\alpha t^q + p\beta t^p)\,d\theta;$$

for in this form the equation shows that the normal is parallel to the stroke $q\alpha t^q + p\beta t^p$, whence it readily follows that if the normal meets x'_q and x'_p at x_q and x_p, then

$$x_q = (p-q)\,x'_q/p$$

and

$$x_p = (q-p)\,x'_p/q.$$

These points x_q and x_p lie on fixed concentric circles whose radii will be called a and b; they are the instantaneous centres in the two ways in which the curve is produced by a rolling circle. A table of these two ways is as follows:

Fixed radius.	**Rolling radius.**	**Arm.***
a	$qa/(q-p)$	$qb/(q-p)$,
b	$pb/(p-q)$	$pa/(p-q)$,

or, in a convenient notation,

a	a_q	b_q,
b	b_p	a_p.

In terms of a and b, the equation of the trochoid (1) is

$$(p-q)\,x = pat^q - qbt^p. \tag{2}$$

When $a = b$, the tracing point is in each case on the rolling circle, and the curve is a cycloid, the equation being

$$(p-q)\,x/a = pt^q - qt^p. \tag{3}$$

---

*That is, the distance from the rolling centre to the tracing point. It must be understood that the rolling radii are the radii of curvature at the point of contact; so that, a and b being positive, when p and q have the same sign either

$$a_q < 0,\ b_p > b \quad \text{or} \quad a_q > a,\ b_p < 0,$$

and when p and q have different signs,

$$0 < a_q < a \quad \text{and} \quad 0 < b_p < b.$$

In the former case the curve is an epitrochoid, in the latter, a hypotrochoid; in accord with Proctor's Geometry of Cycloids.

§2. The method of generating cycloids given by Wolstenholme (Proc. London Math. Soc., t. IV, 1871–3) and Kiepert (Schlömilch's Zeitschrift, t. XVII, 1872) can be shown to be a form of the epicyclic method. For draw through x any line making intercepts α', β' on those sides of the parallelogram which meet at 0, so that $\alpha/\alpha' + \beta/\beta' = 1$. Thus x is a point which divides in a fixed ratio a stroke whose ends describe fixed circles with constant velocities. The radii of these circles are subject only to the equation just written, and we can take, instead of the two, a single circle of radius $\alpha' = \alpha + \beta$. Thus, if two points describe the same circle (α') with velocities p, q, the locus of a point which divides the stroke in a fixed ratio α/β is a trochoid. As before, the normal is parallel to a line from the origin which divides the stroke in the ratio $q\alpha/p\beta$. When $q\alpha = -p\beta$, the stroke is normal to the curve, and when $q\alpha = p\beta$ the stroke is tangent to the curve; in both cases the curve is a cycloid. The second case is the mode of generation of Wolstenholme and Kiepert. It will be noticed that we have here a very simple proof of the fact that the evolute of a cycloid is a cycloid.

We may take instead of the circle (α') a circle (β') of radius $\beta' = \alpha - \beta$; the two circles (α') and (β') are the vertex circles of the trochoids, for evidently the greatest and least values of $|x|$, or

$$|\alpha t^q + \beta t^p|,$$

are $|\alpha \pm \beta|$.

The two tracing circles through x intersect again at a point y. The locus of y is readily determined, for from Fig. A, the points 0, x'_p, y, x'_q form a contra-parallelogram,* and hence, by a calculation which presents no difficulty,

$$y = (\alpha^2 - \beta^2)/(\alpha t^{-q} - \beta t^{-p});$$

therefore the locus of y is the inverse of a trochoid with regard to its centre.

§3. We suppose henceforth that p and q are integers; thus the curves are algebraic.

It is evident that when x_p, $= bt^p$, is given, there are p points of the trochoid corresponding to the p values of t, the normals at these points all passing through x_p. Moreover, from (2), these p points form a regular polygon Γ_p, of circum-radius a_p, concentric with and rigidly attached to the rolling circle (b_p).

---

* I take this term from the works on Linkages; see for example Kempe, How to draw a Straight Line.

In the second mode of trochoidal generation, we must then regard the curve as simultaneously described by the groups of points Γ_p.*

When x_q, $= at^q$, is given, there are q points Γ_q, which play a similar part in the first mode of trochoidal generation.

If we desire to show the simultaneous generation by p or q points, in the equation which defines the curve, all we have to do is to write in (1) $t^p = \tau$, so that

$$(p-q)\,x = pa\tau^{q/p} - qb\tau\,,$$

or $t^q = \tau$, so that

$$(p-q)\,x = pa\tau - qb\tau^{p/q}.$$

But a more symmetrical procedure is to write $t^{pq} = \tau$, so that

$$(p-q)\,x = pa\tau^{1/p} - qb\tau^{1/q}. \tag{4}$$

In this form we take on the circle (a) the p points $a\tau^{1/p}$, and on the circle (b) the q points $b\tau^{1/q}$; and the pq lines joining the one set to the other are all normals to the curve.

In the case of the cycloid we have an evident way of correlating two groups Γ_p and Γ_q, namely, by taking the feet of the $p+q$ normals from any point of the cusp-circle. For the trochoid there is no evident useful way, and herein lies a point of departure. The next four articles refer only to the cycloid; the remaining ones to the trochoid.

§4. In the theory of the cycloid, when we speak of groups Γ_p and Γ_q on the same curve, it will be understood that they are the feet of the normals from any one point of the cusp-circle.

The cycloid whose equation is (3) will, however displaced, be denoted by $C^a_{p,q}$, or (when the size is not in question) by $C_{p,q}$.

A cycloid with all its cusps at Γ_p or Γ_q may be said to be *adjustable* to $C^a_{p,q}$; and when of two cycloids each is adjustable to the other, it may be said that the two are adjustable. The existence of two adjustable cycloids is clear to begin with from the example already stated, in which the rod $C^{2a}_{1,-1}$ has its cusps on the cardioid $C^a_{2,1}$, and conversely, the cardioid has its cusp on the rod so soon as we notice that the cusps of the rod form a group Γ_2 of the cardioid; that is, so soon as we know that the instantaneous centre of rotation of the rod is on the cusp-circle of the cardioid.

---

*This idea is given by Kiepert for the cycloid when $q=1$; by Wolstenholme for any cycloid. I have not come across the evident extension to trochoids.

We have seen that $C^a_{p,\,q}$ is generated by Γ_p, a regular polygon fixed in the rolling circle of radius $a_p \equiv pa/(p-q)$; or by Γ_q, a regular polygon fixed in the rolling circle of radius $a_q \equiv qa/(q-p)$. Let a curve $C^{a_p}_{p_1,\,p_1-p}$ be described with its p cusps at Γ_p; we require the envelope of this curve as the circle (Γ_p) rolls on the fixed circle. From the point of contact x' draw the normals to $C_{p_1,\,p_1-p}$. The feet of these normals form two regular polygons Γ_{p_1} and Γ_{p_1-p}; attending only to the first of these, it lies on a circle of radius $p_1 a_p/p$ or $p_1 a/(p-q)$, which has rolling motion on (a_p) and therefore also on (a). Therefore the locus of Γ_{p_1}, which is part of the envelope of $C^{a_p}_{p_1,\,p_1-p}$, is the curve $C^a_{p_1,\,p_1-p+q}$, or $C^a_{p_1,\,q_1}$, where $p_1-q_1=p-q$. The cusps of this new curve are evidently the cusps of $C^a_{p,\,q}$. Hence we have the general theorem:

Let $C^a_{p,\,q}$ and $C^a_{p_1,\,q_1}$ have the same cusps, so that $p-q=p_1-q_1$. Let the feet of the normals from a point of the common cusp-circle be Γ_p, Γ_q; Γ_{p_1}, Γ_{q_1}. Then the curve $C^{a_p}_{p_1,\,p_1-p}$ with cusps at Γ_p will touch the second curve at Γ_{p_1}. And in this statement it is sufficiently implied also that

$$C^{a_{p_1}}_{p,\,p-p_1} \textit{ with cusps at } \Gamma_{p_1} \textit{ will touch at } \Gamma_p,$$

together with three other pairs of statements of the same kind obtained from these by interchanging p with q, or p_1 with q_1, or p with p_1 and q with q_1.

In applying this result to special cases it is essential to notice what happens when p and q have a common integer factor r. Let us see what is meant by $C_{2,\,-2}$. Geometrically, we say that Γ_2 is inscribed in a circle of radius $a/2$, which rolls with like curvature on a circle of radius a; and the locus is two orthogonal diameters, say $C_{1,\,-1}$ and $iC_{1,\,-1}$. And analytically the same comes from the equation (4), which here becomes

$$4x/a = 2\,(\tau^{\frac{1}{2}} + \tau^{-\frac{1}{2}}),$$

for, according as the product of the roots selected is ± 1, the equation is $x = \pm a \cos\theta/2$ or $x = \pm ia \sin\theta/2$. And so generally by $C_{p,\,q}$, when p and q have the factor r, we understand r curves $C_{p/r,\,q/r}$ obtained from any one curve by successive rotations through the angle $2\pi/(p-q)$.*

---

* It may be the more natural view that when a circle rolls on another, the locus of a polygon Γ_p inscribed in the rolling circle is p distinct curves which may happen to coalesce. But the view taken, namely that the locus is one curve which may happen to break up, is permissible; and it alone renders possible a clear statement of a theorem which covers all the facts in question.

§5. Case 1. Let $p_1 = -q$, $q_1 = -p$. Then C_{p_1, q_1} and $C_{p,q}$ are the same curve, and Γ_{p_1}, Γ_{q_1} coincide respectively with Γ_q, Γ_p. Hence the curve $C^{a_p}_{p+q,q}$ with cusps at Γ_p will touch at Γ_q. Similarly the curve $C^{a_q}_{p,p+q}$ with cusps at Γ_q will touch $C_{p,q}$ at Γ_p; thus if the cusps of $C^{\alpha}_{p,p+q}$ are points Γ_q of $C^{\beta}_{p+q,q}$, then the cusps of $C^{\beta}_{p+q,q}$ are points Γ_p of $C^{\alpha}_{p,p+q}$, provided $\alpha/a_q = \beta/a_p$, or $p\alpha + q\beta = 0$.

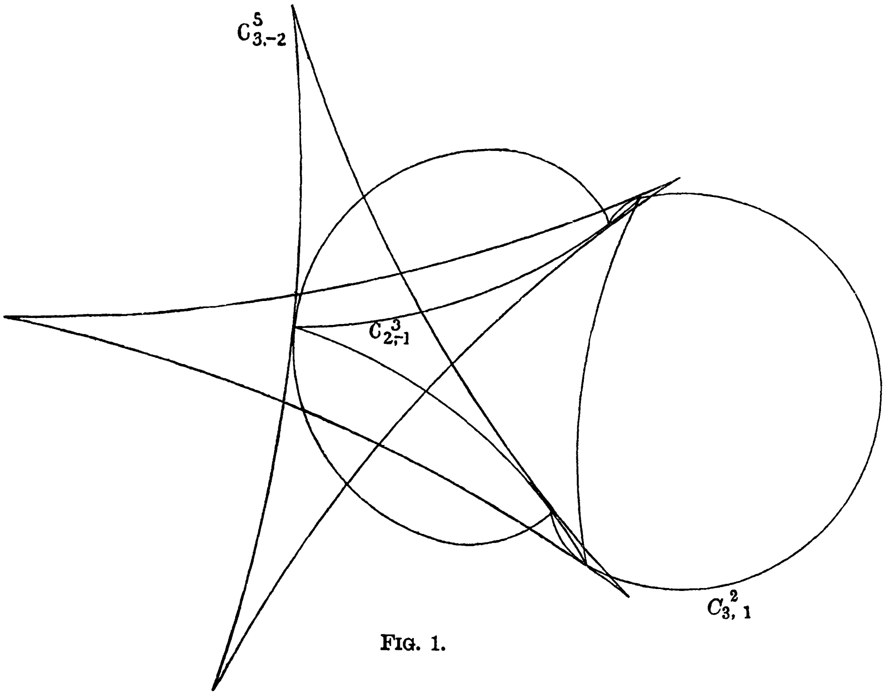

FIG. 1.

See Fig. 1, where $p = 3$, $q = -2$; also Fig. 3, where $p = 3$, $q = -1$; but in Fig. 3 the curve $C_{p,p+q}$ is not drawn.

The above conditions for an adjustable pair of cycloids may be stated as follows:

Let a circle (α), rolling on a circle (β), give the irreducible cycloid $C^{\beta}_{p',q'}$; and let the circle (β), rolling on the circle (α), give the irreducible cycloid $C^{\alpha}_{p,q}$; then the two cycloids are adjustable. For we have

$$\alpha = p'\beta/(p'-q') \text{ or } q'\beta/(q'-p'),$$
$$\beta = p\alpha/(p-q) \text{ or } q\alpha/(q-p),$$

and therefore we have an equation of the form

$$p'/(p'-q') = (p-q)/p,$$

and bearing in mind that a simultaneous change of sign in p and q is permissible, we can say that

$$p' = q - p,$$

$$p' - q' = -p,$$

that is,

$$q = q' = p + p',$$

where

$$p\alpha + p'\beta = 0;$$

and these conditions tally with the above ones.

Case 2. Let $p_1 = -q_1 = (p - q)/2$. Then C_{p_1, q_1} is p_1 equiangular diameters of a circle, and Γ_{p_1} or Γ_{q_1} is the feet of the perpendiculars on these diameters from any point of the circle. If with these feet as cusps a curve C_{p_0, q_0} be described, then writing

$$p = p_0, \; p - p_1 = q_0,$$

so that $p + q = 2q_0$, $q = 2q_0 - p_0$, $p_1 = p_0 - q_0$, $a_{p_1} = a/2$, we have the result that the curve $C^{a}_{p_0, q_0}$ touches $C^{2a}_{p_0, 2q_0 - p_0}$, whose cusp-tangents are the p_1 diameters, at Γ_{p_0}; and similarly it touches $C^{2a}_{q_0, 2p_0 - q_0}$ at Γ_{q_0}.

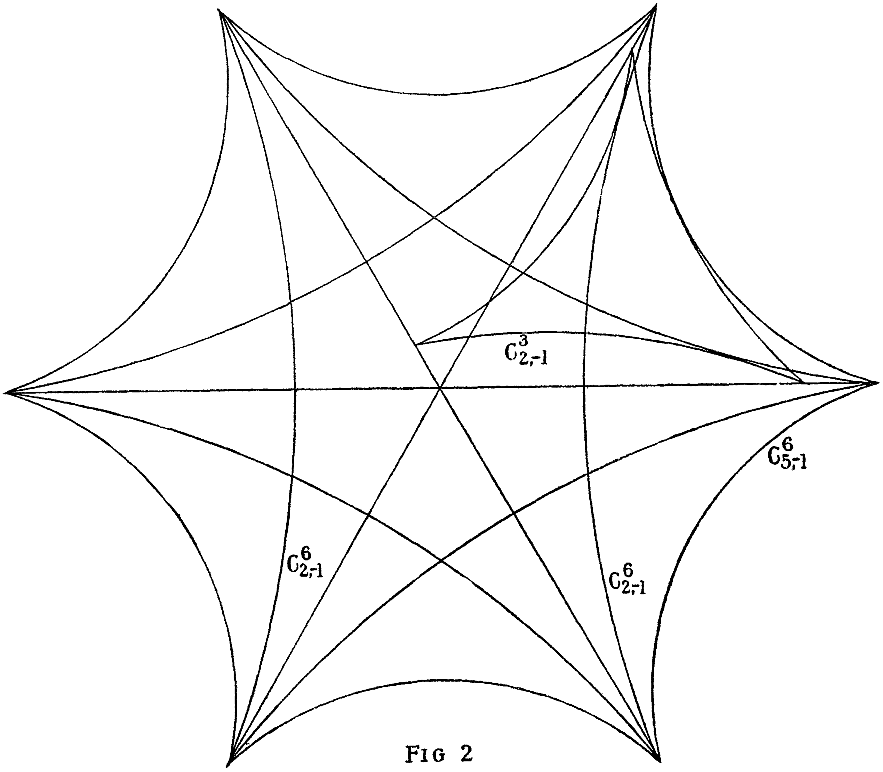

FIG 2

This case is illustrated in Fig. 2, where $p_0 = 2$, $q_0 = -1$; here, as always when p_0 or q_0 is even, one partial envelope breaks up into two curves.

Case 3. As in case 2, let $p_1 = -q_1$. Let a curve C_{p_0, q_0} touch the p_1 diameters at the feet of the perpendiculars from any point of the fixed circle; then if we write

$$p_1 = p_0, \; p_1 - p = q_0,$$

so that $p - q = 2p_0$, $p + q = -2q_0$, $a_p = (p_0 - q_0)a/2p_0$, we see that the cusps of $C^{(p_0-q_0)a}_{p_0, q_0}$ lie on the fixed curve $C^{2p_0 a}_{p_0+q_0, q_0-p_0}$. This is illustrated by the curves $C^3_{2,-1}$ and $C^4_{3,-1}$ in Fig. 3.*

§6. Fig. 3 is an example of a chain of cycloids, each of which is adjustable to the next after it, while all touch at a group Γ_p. The figure shows that the hypocycloids $C^2_{1,-1}$, $C^3_{1,-2}$, ... $C^r_{1,1-r}$ all touch at the single point Γ_1, and each passes through the cusps of the preceding. It will be noticed that the cusps of $C_{1,1-r}$ lie on the cusp-tangents of $C_{1,1-2r}$; but it should be added that this is a peculiarity of the chain selected. For generally if $C_{p,q}$, $C_{p,q-p}$, ... $C_{p,q-rp}$ touch at Γ_p, a necessary condition that the cusps of $C_{p,q}$ lie on the cusp-tangents of $C_{p,q-rp}$ is (see case 2 above)

$$2q - p = q - rp,$$

or

$$q = (1 - r)p,$$

so that $p = \pm 1$, either sign leading to the result that $C_{1,1-r}$, if of the right size, has its cusps on the cusp-tangents of $C_{1,1-2r}$. The proof that $C_{1,1-r}$ is of the right size, in the chain shown in Fig. 3, is immediate, for its cusp-radius is half that of $C_{1,1-2r}$.

It may be remarked that the cusp-circles of all the curves of Fig. 3 must touch at a point lying on the common normal at Γ_1; further, the equation of the curve $C_{1,1-r}$ can be written

$$x = (1 - r)t_0 + rt + t_0^{r-1}/t^r, \tag{5}$$

---

* With respect to the relations of these curves, it is worth remarking that if parallel rays of light fall on $C_{2,-1}$, their direction being those of the common tangent, the caustic by reflexion at $C_{2,-1}$ is $C_{3,-1}$. A proof of this will be found in my note "On the Caustic of the Epicycloid" in the Haverford College Studies, No. 4.

where t_0 is a fixed point and t a variable point on the unit circle, and r is an integer positive or negative. And starting *ab initio* with such a system of equations, we can verify analytically the facts which the figure shows.

§7. The general theorem solves the problems of finding (1) the envelope of a cycloid which is adjustable to a given cycloid, and (2) the cusp-locus of a cycloid which touches a given cycloid at a group Γ_p. There remains the problem of finding the rest of the envelope of a cycloid which touches a given cycloid at Γ_p. This is in reality solved already, for we can find first the cusp-locus, and then the envelope of the curve whose cusp-locus is now known. One part of the envelope is the given curve, and the other part is the curve sought. But it is desirable to eliminate the cusp-locus and express the result solely in terms of the given partial envelope and the moving curve, as follows. From the general theorem,

$C^{ap_1}_{p,p-p_1}$ with cusps Γ_{p_1} on $C^{a}_{p_1,q_1}$ will touch $C^{a}_{p,p-p_1+q_1}$ at Γ_p,
and $C^{ap_1}_{q,q-p_1}$ " " " " " " " $C^{a}_{q,q-p_1+q_1}$ at Γ_q.

To make the curves $C_{p,p-p_1}$ and $C_{q,q-p_1}$ coincide let

$$p+q=p_1.$$

Then $C^{ap_1}_{p,-q}$, which is adjustable to $C^{a}_{p_1,q_1}$, has the partial envelopes C^{a}_{p,q_1-q} and C^{a}_{q,q_1-p}.

Write $-q$ for q, and $p-r$ for q_1-q; then $a_{p_1}=(p-q)\,a/r$; thus:

If $C^{(p-q)a}_{p,q}$ touches a fixed curve $C^{ra}_{p,p-r}$ at Γ_p, it touches the fixed curve $C^{ra}_{q,q+r}$ at Γ_q, the two fixed curves having the same cusps.

For example, let $r=2p$; then the curve $C^{(p-q)a}_{p,q}$ slides on p equiangular lines, and envelopes the curve $C^{2pa}_{q,q+2p}$. This is illustrated by the curves $C^{3}_{2,-1}$ and $C^{4}_{3,-1}$ of Fig. 3.

§8. In the cycloidal case, when $a=b$, the cusps are the most salient points, and accordingly the theory was explained with reference to them alone. But we might have selected any definite group of $(p-q)$ points, forming a regular polygon concentric with the curve, instead of the cusps; and in dealing with the trochoid it is not advisable to specify the group beforehand. Only, for simplicity, we suppose the group on the chief axes of the curve, that is on those axes which are cusp-tangents when $a=b$. Such a group of $p-q$ points, which is a group *of* the curve but not necessarily *on* the curve, will be called a *chief* group.

A chief group is evidently given by any one of its points—say a point on the real axis. When, for two trochoids given in the form (6), the two points which indicate two chief groups are given by the same value of t, we can say that the chief group of the one curve *corresponds* to the chief group of the other. A trochoid T_1 is adjustable to a trochoid T, when a definite chief group of T_1 is a group Γ_p or Γ_q of T; and the two trochoids are adjustable when also the corresponding chief group of T is a group Γ_{p_1} or Γ_{q_1} of T_1.

We denote the trochoid given by equation (2), however displaced, by $T^{a,\,b}_{p,\,q}$.

Consider the group Γ_p which is describing the curve $T^{a,\,b}_{p,\,q}$. With Γ_p as a chief group describe a curve $T^{a_0,\,b_0}_{p_0,\,q_0}$, which moves with the rolling circle (b_p). This involves

$$p_0 - q_0 = p, \tag{8}$$

and a further condition, expressed by the equation

$$(p_0 a_0 t^{q_0} - q_0 b_0 t^{p_0})/(p_0 - q_0) = a_p \tag{9}$$

and its conjugate.

Further, let us specialize by making the circle (b_p) a fixed circle of $T^{a_0,\,b_0}_{p_0,\,q_0}$; that is, let

$$b_0 = pb/(p-q). \tag{10}$$

The feet of the normals from the point of contact of the circles (b) and (b_0) to $T^{a_0,\,b_0}_{p_0,\,q_0}$ now contain a group Γ_{p_0}; this group has the circum-radius a_{p_0} and is attached concentrically to a circle (b_{p_0}) which touches the circles (b) and (b_0) at their point of contact. Since (b_{p_0}) rolls on (b_0) while (b_0) rolls on (b), (b_{p_0}) rolls on (b). Therefore the locus of Γ_{p_0} is a trochoid $T^{a_1,\,b_1}_{p_1,\,q_1}$ determined by the equations

$$p_1 = p_0, \tag{11}$$

$$b_1 = b, \tag{12}$$

$$p_1 b_1/(p_1 - q_1) = p_0 b_0/(p_0 - q_0), \tag{13}$$

$$p_1 a_1/(p_1 - q_1) = p_0 a_0/(p_0 - q_0). \tag{14}$$

This trochoid is, of course, only part of the envelope of $T^{a_0,\,b_0}_{p_0,\,q_0}$; how to ascerta the remainder is another question.

From (8), (11), (13),

$$b_1/(p_1 - q_1) = b_0/p,$$

and therefore, from (10) and (12),

$$p_1 - q_1 = p - q, \ = r \text{ suppose.}$$

Equations (8), (11), (14) give

$$a_0 = pa_1/r.$$

We have thus the result that if we take two trochoids $T^{a,\,b}_{p,\,p-r}$ and $T^{a_1,\,b}_{p_1,\,p_1-r}$ with coincident chief axes, and groups Γ_p and Γ_{p_1} on them, the normals at which all meet on the common fixed circle (b), then a trochoid $T^{pa_1/r,\,pb/r}_{p_1,\,p_1-p}$, with a chief group at Γ_p, will touch at Γ_{p_1}; the possibility of drawing the third trochoid, subject to the conditions, being insured by a certain relation between the constants, which relation is included in that obtained by eliminating t from

$$pa = p_1 a_1 t^{p_1 - p} - (p_1 - p)\, bt^{p_1} \tag{15}$$

and the conjugate relation.

Similarly a trochoid $T^{p_1 a/r,\,p_1 b/r}_{p,\,p-p_1}$, with a chief group at Γ_{p_1}, will touch at Γ_p; the existence of this fourth trochoid being insured by a certain relation.

But this relation depends on the elimination of t from the equation

$$p_1 a_1 = pat^{p-p_1} - (p - p_1)\, bt^{p} \tag{16}$$

and its conjugate; and the equations (15) and (16) are the same, either being

$$(p - p_1)\, b = pat^{-p_1} - p_1 a_1 t^{-p}. \tag{17}$$

§9. If, then, we select a value of t from the equation (17), and thereby a chief group of $T^{pa_1/r,\,pb/r}_{p_1,\,p_1-p}$, and the *corresponding* chief group of $T^{p_1 a/r,\,p_1 b/r}_{p,\,p-p_1}$, we have the theorem that if the chief group of the former is a group Γ_p on the latter, the chief group of the latter is a group Γ_{p_1} on the former.

We are thus led to study especially the relation of the two trochoids

$$T_1 \equiv T^{p\alpha_1,\,p\beta}_{p_1,\,p_1-p} \text{ and } T \equiv T^{p_1\alpha,\,p_1\beta}_{p,\,p-p_1},$$

the factor $1/r$ being now included in α, α_1, β; and here an analytical method is convenient. All that will be established for trochoids matches what was given in case 1 of §5 for two cycloids; though the preceding article is appropriate as pointing to developments not now attempted, parallel to the rest of §5.

The two curves are produced as in the table :

	Fixed radius.	Rolling radius.	Arm.
T	$p_1\beta$	$p\beta$	$p\alpha$,
T_1	$p\beta$	$p_1\beta$	$p_1\alpha_1$.

The equations are

$$\left.\begin{aligned} &\text{for } T, && x = p\alpha t^{p-p_1} - (p-p_1)\beta t^p, \\ &\text{for } T_1, && x_1 = p_1\alpha_1 t_1^{p_1-p} - (p_1 - p)\beta t_1^{p_1}. \end{aligned}\right\} \qquad (18)$$

The curve T admits a group Γ_p, whose mean point is $-(p-p_1)\beta t^p$. Let the second curve experience the turn t^{p-p_1},* and let it be placed concentrically with Γ_p; its equation becomes

$$x_1 = -(p-p_1)\beta t^p + \{p_1\alpha_1 t_1^{p_1-p} - (p_1-p)\beta t_1^{p_1}\}\, t^{p-p_1},$$

where t is regarded as constant.

The equation is

$$x_1 = (p-p_1)\beta t^p \{(t_1/t)^{p_1} - 1\} + p_1\alpha_1 (t_1/t)^{p_1-p}.$$

Therefore, when $(t_1/t)^{p_1}$, $\equiv \epsilon_1^{p_1}$, $=1$, $x_1 = p_1\alpha_1\epsilon_1^{-p}$; therefore, if we suppose T_1 rigidly attached to Γ_p, T_1 passes through p_1 fixed points as Γ_p describes T.†

The p_1 points $p_1\alpha_1\epsilon_1^{-p}$ form a chief group of T, and a group Γ_{p_1} on T_1; and conversely, the p points $p\alpha\epsilon^{-p_1}$ (where $\epsilon^p = 1$), referred to the centre of T_1, are a chief group of T_1 and a group Γ_p on T.

§10. Let us take as a first example the chief group of T given by $t^{p_1} = 1$. This is a group of vertices; in fact it is that group of vertices which coincide with the cusps when the fixed radii become equal, if these fixed radii have the same sign. These points of T are a group Γ_{p_1} of T_1 if

$$p_1\alpha_1 = p\alpha - (p-p_1)\beta;$$

that is, if

$$p_1(\alpha_1 - \beta) = p(\alpha - \beta), \qquad (19)$$

---

* More generally any constant turn τ (where $|\tau| = 1$) may be included, when we wish to consider, with regard to $T_{p,q}^{a,b}$, the case of those groups of $(p-q)$ points which do not lie on the chief axes.

† Without mention of the curve T, we can say : Let a circle $(p_1\beta)$ roll on a circle $(p\beta)$, and let T_1 be any algebraic trochoid so formed. Regard T_1 as rigidly attached to $(p\beta)$; then when $(p\beta)$ rolls on $(p_1\beta)$, T_1 passes through p_1 fixed points.

and this symmetrical condition also ensures that when the group of vertices of T is placed on T_1, the corresponding group of vertices of T_1, given by $t_1^p = 1$, shall lie on T. This is shown in Figs. 4 and 5, when $p = 3$, $p_1 = 2$. It will be noticed in Fig. 5 that, α and β being positive and $3\alpha < \beta$, the condition (19) makes α_1 negative, and the vertex given by $t = 1$ is no longer one which coincides with a cusp in the cycloidal case.

To obtain two curves adjustable in this way, by rolling each of two circles on the other, we must take in both cases a tracing point at the same distance from the nearest point of the rolling circle. See equation (19) and the table of §9.

§11. Passing to the nodes, we can distinguish among them, in any trochoid $T^{a,\,b}_{p,\,q}$, those which become cusps when $a = b$; these form a group of $p - q$ nodes which can be called *primary nodes.*

To determine the nodes which lie on a chief axis, we make x real in the equation (6); that is, we seek to solve the equation

$$(p - q)\,n = pat^q - qbt^p = pat^{-q} - qbt^{-p}. \tag{20}$$

These give

$$pa\,(t^q - t^{-q}) = qb\,(t^p - t^{-p}),$$

or passing over the values $t = \pm 1$ (which give vertices of the curve), we have, when p and q are positive,

$$pa\,(t^{q-1} + t^{q-3} + \ldots . + t^{-q+1}) = qb\,(t^{p-1} + t^{p-3} + \ldots . + t^{-p+1}),$$

and, whatever signs p and q have, the last equation holds if for p and q we write $|p|$ and $|q|$. When the equation can be solved, we have all the nodes on the axis considered, and the primary nodes on this axis are those values of n which $= b$ when $a = b$. The remaining primary nodes are obtained by multiplying the ones found by $\exp 2\pi i/(p - q)$.

For example, in the curve $T^{a,\,b}_{3,\,2}$, the equation for $t + t^{-1}$ is

$$3a\,(t + t^{-1}) = 2b\,(t^2 + 1 + t^{-2}),$$

and

$$n = 3at^2 - 2bt^3$$
$$= 2b\,(t + t^{-1}) - 3a;$$

therefore

$$2n = -3a \pm \sqrt{9a^2 + 16b^2},$$

and the upper sign gives the primary node.

It will be observed that the determination of the nodes of the two curves $T^{a,\,b}_{p,\,\pm q}$ depends on the same equation.

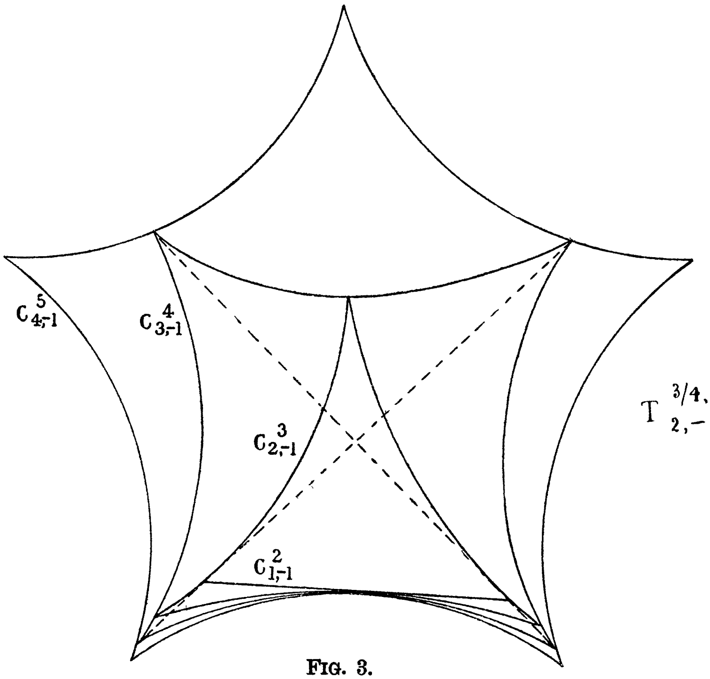

FIG. 3.

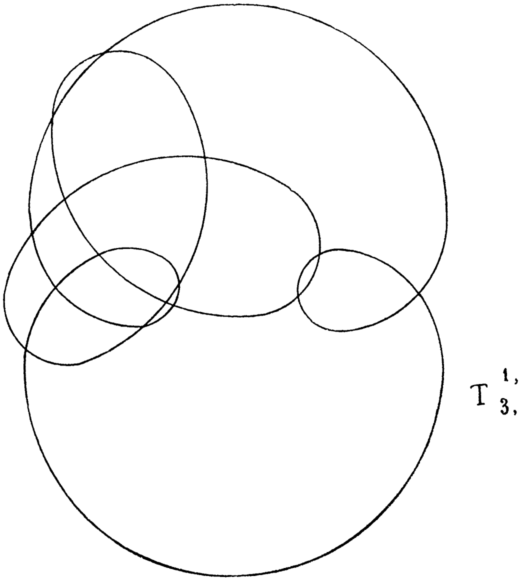

FIG. 4.

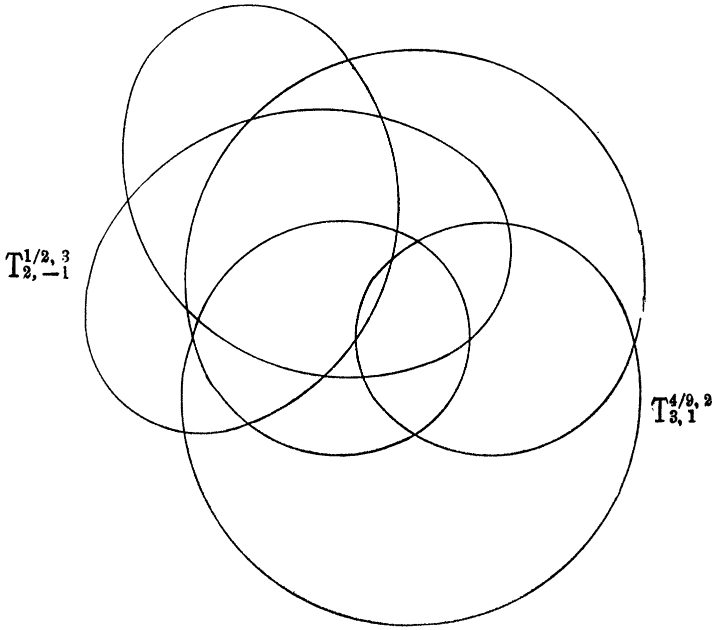

FIG. 5.

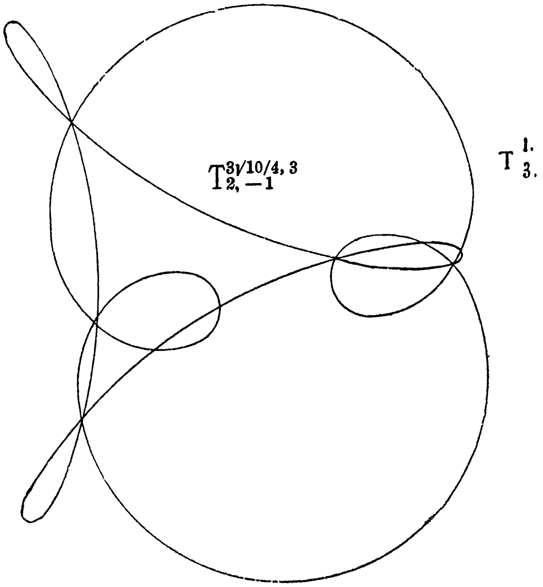

FIG. 6.

The actual results in some simple cases are

$$\left.\begin{aligned}
n^{a,\,b}_{2,\,1} &= b, \\
3n^{a,\,b}_{2,\,-1} &= (4a^2 - b^2)/b, \\
2n^{a,\,b}_{3,\,1} &= \pm\sqrt{b\,(3a+b)}, \\
4n^{a,\,b}_{3,\,-1} &= \pm\,(3a-b)\sqrt{(3a+b)/b}, \\
2n^{a,\,b}_{3,\,2} &= -\,3a + \sqrt{9a^2+16b^2}, \\
10n^{a,\,b}_{3,\,-2} &= \left(\frac{9a^2}{4b^2} - 1\right)(3a + \sqrt{9a^2+16b^2}).
\end{aligned}\right\} \tag{21}$$

So far, at all events, the primary nodes are crunodes when $a < b$.

Now, returning to the two trochoids T and T_1 of §9, the equation

$$p_1\alpha_1 = p\alpha t^{p-p_1} - (p-p_1)\,\beta t^p;$$

that is,

$$(p-p_1)\,\beta = p\alpha t^{-p_1} - p_1\alpha_1 t^{-p}, \tag{17^{\text{bis}}}$$

and its conjugate, express that the chief group of T, which lies on T_1, is a group *on* T, and equally express that the chief group of T_1, which lies on T, is a group *on* T_1. But it is not easy to analytically discriminate the group of primary nodes from the other groups of nodes on the chief axes; while without such a discrimination we are not entitled to infer that the primary nodes of T and T_1 are corresponding groups, merely from the fact that they do correspond when $\alpha = \alpha_1 = \beta$.

There is, of course, no difficulty in making a precise statement when we can explicitly obtain the nodes. Let us take, as before, $p = 3$, $p_1 = 2$. The above equation (17^{bis}) is now

$$n^{\alpha,\,\alpha_1}_{3,\,2} = \beta,$$

or from (21),

$$(2\beta + 3\alpha)^2 = 9\alpha^2 + 16\alpha_1^2;$$

and this equation, written in the form

$$\pm\sqrt{2\beta\,(6\alpha + 2\beta)} = 2.2\alpha_1,$$

expresses that the primary nodes of $T^{2\alpha,\,2\beta}_{3,\,1}$ can be a group T_2 of $T^{3\alpha_1,\,3\beta}_{2,\,-1}$, while, written in the form

$$(4.9\alpha_1^2 - 9\beta^2)/3\beta = 3.3\alpha,$$

it expresses that the primary nodes of the second curve can be a group T_3 of the first (Fig. 6).

§12. The foci of the curve (6) are given by $D_t x = 0$; that is, by

$$at^{q-1} = bt^{p-1}.$$

Disregarding those which are met together at the centre of the curve, we have $p - q$ others which can be called chief foci. Of the curves T and T_1 of §9, if the chief foci of the one lie on the other, then the chief foci of the other lie on the one.

For the chief foci of T are given by

$$\alpha = \beta t^{p_1}$$

and are
$$f = p_1 \beta t^p.$$

These coincide with the fixed points $p_1 \alpha_1 \epsilon_1^{-p}$ of §9 if

$$\beta t^p = \alpha_1;$$

that is, if
$$(\alpha / \beta)^{p_1} = (\alpha_1 / \beta)^p, \tag{22}$$

and this is a symmetrical relation.

§13. Finally, it should be observed that, in all cases, the connexions of the curves considered are consistent with relative motion; in particular, two adjustable curves are capable of relative motion without breach of the adjustment.

HAVERFORD COLLEGE.